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The Continuity Property Via g^{ω}- Open sets in Grill Topological Spaces

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ABSTRACT

In this paper, we introduce introduces and investigates the notion of $\mathcal{G}^{\omega}$-continuous functions via class of $\mathcal{G}\beta$-open sets and we study θ-cluster operator via this class to introduces and investigates the notion of $\theta-\mathcal{G}^{\omega}$-continuous functions in grill topological spaces. The relationships between the pervious functions and other known functions are introduced and studied.

AMS classification: Primary 54C08, 54C05.

Keywords

open sets; Grill; Grill topological spaces.

1. INTRODUCTION

The continuity property is one of the fundamental concepts in point-set topology. In 1982 Hdeib [5], introduced the notion of ω-open set and ω-continuous function as a weaker form of open set and continuous function, respectively, in topological spaces. A subset A of a space (X,τ) is called *ω-open set* if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. A function $f : (X,\tau) \to (Y,\rho)$ of a topological space (X,τ) into a topological space (Y,ρ) is called *ω-continuous function* if for each $x \in X$ and for an open set G in Y containing $f(x)$, there is ω-open set U in X containing x such that $f(U) \subseteq G$. In 1983 [7] introduced the notion of β-open set and β-continuous function which are two of the famous weak forms of open set and continuous function, respectively, in topological spaces. A subset A of a space (X,τ) is called *β-open set* if $A \subseteq Cl(Int(Cl(A)))$. A function $f : (X,\tau) \to (Y,\rho)$ is *β-continuous function* if $f^{-1}(U)$ is β-open set in X for every open set U in Y. Under the notions of ω-open sets and β-open sets, [9] introduced the notion of $\beta\omega$-open set as a weak form for ω-open sets and β-open sets.

For the study of grill topological spaces, [1] introduced the notions of $\mathcal{G}\beta$-open set and $\mathcal{G}\beta$-continuous function as a strong forms of β-open set and β-continuous function, respectively, in grill topological spaces. A subset A of a grill topological space $(X,\tau,\mathcal{G})$ is called *$\mathcal{G}\beta$-open set* if $A \subseteq Cl(Int(\Psi(A)))$. A function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ of a grill topological space $(X,\tau,\mathcal{G})$ into a space (Y,ρ) is called *$\mathcal{G}\beta$-continuous function* if $f^{-1}(U)$ is $\mathcal{G}\beta$-open set in $(X,\tau,\mathcal{G})$ for every open set U in Y. In [2], we introduced the notion of $\mathcal{G}^{\omega}$-open set as a form stronger than $\beta\omega$-open set and weaker than ω-open set and $\mathcal{G}\beta$-open set. A subset G of grill topological space $(X,\tau,\mathcal{G})$ is called *$\mathcal{G}^{\omega}$-open set* if $G \subseteq Cl(Int_{\omega}(\Psi(G)))$. The complement of $\mathcal{G}^{\omega}$-open set is called *$\mathcal{G}^{\omega}$-closed set*, where $Int_{\omega}(A)$ denotes to ω-interior operator of A which is defined as the union of all ω-open subsets of X contained in A. $Cl_{\omega}(A)$ denotes to ω-closure operator of A which is defined as the intersection of all ω-closed subsets of X containing A.

In this paper, we introduce the continuity property via class of $\mathcal{G}^{\omega}$-open sets in grill topological spaces. This paper is organized as follows. In Section 2, we introduce introduces and investigates the notion of $\mathcal{G}^{\omega}$-continuous functions via class of $\mathcal{G}\beta$-open sets. In Section 3, we study θ-cluster operator via the class of $\mathcal{G}^{\omega}$-open sets to introduces and investigates the notion of $\theta-\mathcal{G}^{\omega}$-continuous functions in grill topological spaces. The relationships between the pervious functions and other known functions are introduced and studied.

By $Cl(A)$ and $Int(A)$ we mean the closure set and the interior set of A in topological space (X,τ), respectively.

DEFINITION 1.1. [8] Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called *θ-cluster point* of A if $Cl(U) \cap A \neq \emptyset$ for every open set U in X containing x.

The set of all θ-cluster points of A is called the *θ-cluster set* of A and denoted by $Cl^{\theta}(A)$. A subset A of topological space is called *θ-closed set* in X, [6], if $Cl^{\theta}(A) = A$. The complement of θ-closed set in X is called *θ-open set* in X.

THEOREM 1.2. [8] Every θ-closed set is closed set.

A collection $\mathcal{G}$ of subsets of a topological spaces (X,τ) is said to be a *grill* [4] on X if $\mathcal{G}$ satisfies the following conditions:

(1) $\emptyset \notin \mathcal{G}$;
(2) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$;
(3) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a grill $\mathcal{G}$ on a topological space X, an operator from the power set $P(X)$ of X to $P(X)$ was defined in [3] in the following manner : For any $A \in P(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for each open neighborhood } U \text{ of } x\}.$$

Then the operator $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [3] to be a Kuratowski closure operator, defining a unique topology $\tau_{\mathcal{G}}$ on X such that $\tau \subseteq \tau_{\mathcal{G}}$. This topology defined by

$$\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\},$$

where $\tau \subseteq \tau_{\mathcal{G}}$ and for any $A \subseteq X$, $\Psi(A) = {}_{\mathcal{G}}Cl(A)$ such that ${}_{\mathcal{G}}Cl(A)$ denotes the set of all closure points of A in topological space $(X, \tau_{\mathcal{G}})$. The set of all interior points of A in topological space $(X, \tau_{\mathcal{G}})$ denoted by ${}_{\mathcal{G}}Int(A)$.
If (X, τ) is a topological space and $\mathcal{G}$ is a grill on X then the triple $(X, \tau, \mathcal{G})$ will be called a *grill topological space*.
The following definitions and theorem are taken from [2].

THEOREM 1.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space. If G_k is $\mathcal{G}^\omega$-open set for each $k \in I$ then $\cup_{k \in I} G_k$ is $\mathcal{G}^\omega$-open set, where I is an index set.

THEOREM 1.4. Let $(X, \tau, \mathcal{G})$ be a grill topological space. If G is an open set in (X, τ) and H is $\mathcal{G}^\omega$-open set then $G \cap H$ is $\mathcal{G}^\omega$-open set.

DEFINITION 1.5. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $G \subseteq X$. The *$\theta-\mathcal{G}^\omega$-cluster operator* of G is defined by the set of all $\theta-\mathcal{G}^\omega$-cluster points of G and denoted by ${}_{\mathcal{G}^\omega}Cl^\theta(G)$. A point $x \in X$ is called *$\theta-\mathcal{G}^\omega$-cluster point* of G if ${}_{\mathcal{G}^\omega}Cl(U) \cap G \neq \emptyset$ for every $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x.

DEFINITION 1.6. A subset G of grill topological space $(X, \tau, \mathcal{G})$ is called *$\theta-\mathcal{G}^\omega$-closed set* in $(X, \tau, \mathcal{G})$ if ${}_{\mathcal{G}^\omega}Cl^\theta(G) = G$. The complement of $\theta-\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ is called *$\theta-\mathcal{G}^\omega$-open set* in $(X, \tau, \mathcal{G})$.

THEOREM 1.7. Every θ-closed set in a space (X, τ) is $\theta-\mathcal{G}^\omega$-closed set in grill topological space $(X, \tau, \mathcal{G})$ and every $\theta-\mathcal{G}^\omega$-closed set is $\mathcal{G}^\omega$-closed set.

2. $\mathcal{G}^\omega$-CONTINUOUS FUNCTIONS

DEFINITION 2.1. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ of a grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) is called *$\mathcal{G}^\omega$-continuous function* if $f^{-1}(U)$ is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ for every open set U in Y.

It is clear that every ω-continuous function is $\mathcal{G}^\omega$-continuous function but the converse of this fact no need to be true.

EXAMPLE 2.2. Let $f : (R, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function defined by

$$f(x) = \begin{cases} a, & x \in R - \{2\} \\ b, & x = 2 \end{cases}$$

where $Y = \{a, b\}$,

$$\tau = \{\emptyset, R, R - \{1\}\},\ \mathcal{G} = P(R) - \{\emptyset\},\ \text{and}\ \rho = \{\emptyset, Y, \{b\}\}.$$

The function f is $\mathcal{G}^\omega$-continuous, since $f^{-1}(\{b\}) = \{2\}$ and $f^{-1}(Y) = R$ are $\mathcal{G}^\omega$-open sets in $(R, \tau, \mathcal{G})$. The function f is not ω-continuous, since $f^{-1}(\{b\}) = \{2\}$ is not ω-open set.

It is clear that every $\mathcal{G}\beta$-continuous function is $\mathcal{G}^\omega$-continuous function but the converse of this fact no need to be true.

EXAMPLE 2.3. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ be a function defined by $f(a) = 2$ and $f(c) = f(b) = 1$, where $X = \{a, b, c\}$, $Y = \{1, 2\}$

$$\tau = \{\emptyset, X, \{a\}\},\ \mathcal{G} = P(X) - \{\emptyset\},\ \text{and}\ \rho = \{\emptyset, Y, \{1\}\}.$$

The function f is $\mathcal{G}^\omega$-continuous, since $f^{-1}(\{1\}) = \{b, c\}$ and $f^{-1}(Y) = X$ are $\mathcal{G}^\omega$-open sets in $(X, \tau, \mathcal{G})$. The function f is not $\mathcal{G}\beta$-continuous, since $f^{-1}(\{1\}) = \{b, c\}$ is not $\mathcal{G}\beta$-open set.

THEOREM 2.4. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ of a grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) is $\mathcal{G}^\omega$-continuous if and only if $f^{-1}(F)$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ for every closed set F in Y.

THEOREM 2.5. If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ is $\mathcal{G}^\omega$-continuous function if and only if for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists $\mathcal{G}^\omega$-open set V in $(X, \tau, \mathcal{G})$ such that $x \in V$ and $f(V) \subseteq U$.

PROOF. Suppose that f is $\mathcal{G}^\omega$-continuous function. Let $x \in X$ and U be any open set in Y containing $f(x)$. Put $V = f^{-1}(U)$. Since f is a $\mathcal{G}^\omega$-continuous then V is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ such that $x \in V$ and $f(V) \subseteq U$.
Conversely, Let U be any open set in Y. For each $x \in f^{-1}(U)$, $f(x) \in U$. Then by the hypothesis, there exists $\mathcal{G}^\omega$-open set V_x in $(X, \tau, \mathcal{G})$ such that $x \in V_x$ and $f(V_x) \subseteq U$. This implies, $V_x \subseteq f^{-1}(U)$ and so $f^{-1}(U) = \cup_{x \in f^{-1}(U)} V_x$. Hence by Theorem (1.3),

$$f^{-1}(U) = \cup_{x \in f^{-1}(U)} V_x$$

is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$. That is, f is $\mathcal{G}^\omega$-continuous. □

THEOREM 2.6. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ is $\mathcal{G}^\omega$-continuous of grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) if and only if

$$f[{}_{\mathcal{G}^\omega}Cl(A)] \subseteq {}_{\rho}Cl(f(A))\ \text{for all}\ A \subseteq X.$$

PROOF. Let f be $\mathcal{G}^\omega$-continuous function and A be any subset of X. Then ${}_{\rho}Cl(f(A))$ is a closed set in Y. Since f is $\mathcal{G}^\omega$-continuous then by Theorem (2.4), $f^{-1}[{}_{\rho}Cl(f(A))]$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. That is,

$${}_{\mathcal{G}^\omega}Cl\big[f^{-1}[{}_{\rho}Cl(f(A))]\big] = f^{-1}[{}_{\rho}Cl(f(A))].$$

Since $f(A) \subseteq {}_{\rho}Cl(f(A))$ then $A \subseteq f^{-1}[{}_{\rho}Cl(f(A))]$. This implies,

$${}_{\mathcal{G}^\omega}Cl(A) \subseteq {}_{\mathcal{G}^\omega}Cl\big[f^{-1}[{}_{\rho}Cl(f(A))]\big] = f^{-1}[{}_{\rho}Cl(f(A))].$$

Hence $f[{}_{\mathcal{G}^\omega}Cl(A)] \subseteq {}_{\rho}Cl(f(A))$.
Conversely. let H be any closed set in Y, that is, ${}_{\rho}Cl(H) = H$. Since $f^{-1}(H) \subseteq X$. Then by the hypothesis,

$$f\big[{}_{\mathcal{G}^\omega}Cl[f^{-1}(H)]\big] \subseteq {}_{\rho}Cl[f(f^{-1}(H))] \subseteq {}_{\rho}Cl(H) = H.$$

This implies, ${}_{\mathcal{G}^\omega}Cl[f^{-1}(H)] \subseteq f^{-1}(H)$. Hence ${}_{\mathcal{G}^\omega}Cl[f^{-1}(H)] = f^{-1}(H)$, that is, $f^{-1}(H)$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. Hence by Theorem (2.4), f is $\mathcal{G}^\omega$-continuous. □

THEOREM 2.7. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \rho)$ is $\mathcal{G}^\omega$-continuous of grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) if and only if

$${}_{\mathcal{G}^\omega}Cl(f^{-1}(B)) \subseteq f^{-1}({}_{\rho}Cl(B))\ \text{for all}\ B \subseteq Y.$$

PROOF. Let f be $\mathcal{G}^\omega$-continuous function and B be any subset of Y. Then ${}_{\rho}Cl(B)$ is a closed set in Y. Since f is $\mathcal{G}^\omega$-continuous then by Theorem (2.4), $f^{-1}[{}_{\rho}Cl(B)]$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. That is,

$${}_{\mathcal{G}^\omega}Cl\big[f^{-1}[{}_{\rho}Cl(B)]\big] = f^{-1}[{}_{\rho}Cl(B)].$$

Since $B \subseteq {}_{\rho}Cl(B)$ then $f^{-1}(B) \subseteq f^{-1}[{}_{\rho}Cl(B)]$. This implies,

$${}_{\mathcal{G}^\omega}Cl(f^{-1}(B)) \subseteq {}_{\mathcal{G}^\omega}Cl\big[f^{-1}[{}_{\rho}Cl(B)]\big] = f^{-1}[{}_{\rho}Cl(B)].$$

Hence ${}_{\mathcal{G}^\omega}Cl(f^{-1}(B)) \subseteq f^{-1}[{}_{\rho}Cl(B)]$.

Conversely, let H be any closed set in Y, that is, ${}_\rho Cl(H) = H$. Since $H \subseteq Y$. Then by the hypothesis,

$$_{\mathcal{G}^\omega}Cl(f^{-1}(H)) \subseteq f^{-1}({}_\rho Cl(H)) = f^{-1}(H).$$

This implies, $_{\mathcal{G}^\omega}Cl[f^{-1}(H)] \subseteq f^{-1}(H)$. Hence $_{\mathcal{G}^\omega}Cl[f^{-1}(H)] = f^{-1}(H)$, that is, $f^{-1}(H)$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. Hence by Theorem (2.4), $f^{-1}(H)$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. That is, f is $\mathcal{G}^\omega$-continuous. □

THEOREM 2.8. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is $\mathcal{G}^\omega$-continuous of grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) if and only if

$$f^{-1}({}_\rho Int(B)) \subseteq {}_{\mathcal{G}^\omega}Int[f^{-1}(B)] \text{ for all } B \subseteq Y.$$

PROOF. Let f be $\mathcal{G}^\omega$-continuous function and B be any subset of Y. Then ${}_\rho Int(B)$ is an open set in Y. Since f is $\mathcal{G}^\omega$-continuous then $f^{-1}[{}_\rho Int(B)]$ is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$. That is,

$$_{\mathcal{G}^\omega}Int\big[f^{-1}[{}_\rho Int(B)]\big] = f^{-1}[{}_\rho Int(B)].$$

Since ${}_\rho Int(B) \subseteq B$ then $f^{-1}[{}_\rho Int(B)] \subseteq f^{-1}(B)$. This implies,

$$f^{-1}[{}_\rho Int(B)] = {}_{\mathcal{G}^\omega}Int\big[f^{-1}[{}_\rho Int(B)]\big] \subseteq {}_{\mathcal{G}^\omega}Int(f^{-1}(B)).$$

Hence $f^{-1}({}_\rho Int(B)) \subseteq {}_{\mathcal{G}^\omega}Int[f^{-1}(B)]$.
Conversely, let U be any open set in Y, that is, ${}_\rho Int(U) = U$. Since $U \subseteq Y$. Then by the hypothesis,

$$f^{-1}(U) = f^{-1}({}_\rho Int(U)) \subseteq {}_{\mathcal{G}^\omega}Int[f^{-1}(U)].$$

This implies, $f^{-1}(U) \subseteq {}_{\mathcal{G}^\omega}Int[f^{-1}(U)]$. Hence $f^{-1}(U) = {}_{\mathcal{G}^\omega}Int[f^{-1}(U)]$, that is, $f^{-1}(U)$ is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$. Hence f is $\mathcal{G}^\omega$-continuous. □

DEFINITION 2.9. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ of a grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) is called a *$\mathcal{G}^\omega$-closed function* if $f(G)$ is a closed set in (Y, ρ) for every $\mathcal{G}^\omega$-closed set G in $(X, \tau, \mathcal{G})$.

THEOREM 2.10. Let $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ and $h : (Y, \rho) \to (Z, \gamma)$ be two functions. Then $h \circ f$ is $\mathcal{G}^\omega$-closed function if h is a closed function and f is $\mathcal{G}^\omega$-closed function

PROOF. Let U be $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. Since f is $\mathcal{G}^\omega$-closed function then $f(U)$ is a closed set in Y. Since h is closed function then $h[f(U)] = (h \circ f)(U)$ is That is, $h \circ f$ is a $\mathcal{G}^\omega$-closed function. □

THEOREM 2.11. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is a $\mathcal{G}^\omega$-closed function if and only if ${}_\rho Cl[f(A)] \subseteq f[{}_{\mathcal{G}^\omega}Cl(A)]$ for all $A \subseteq X$.

PROOF. Suppose that f is $\mathcal{G}^\omega$-closed function and A be any subset of X. Since $_{\mathcal{G}^\omega}Cl(A)$ is $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ and f is $\mathcal{G}^\omega$-closed function then $f[{}_{\mathcal{G}^\omega}Cl(A)]$ is a closed set in Y. That is,

$$_\rho Cl\big[f[{}_{\mathcal{G}^\omega}Cl(A)]\big] = f[{}_{\mathcal{G}^\omega}Cl(A)].$$

Since $A \subseteq {}_{\mathcal{G}^\omega}Cl(A)$ then $f(A) \subseteq f[{}_{\mathcal{G}^\omega}Cl(A)]$. This implies,

$$_\rho Cl[f(A))] \subseteq {}_\rho Cl\big[f[{}_{\mathcal{G}^\omega}Cl(A)]\big] = f[{}_{\mathcal{G}^\omega}Cl(A)].$$

Hence ${}_\rho Cl[f(A)] \subseteq f[{}_{\mathcal{G}^\omega}Cl(A)]$.
Conversely, let F be any $\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$, that is, $_{\mathcal{G}^\omega}Cl(F) = F$. Since $F \subseteq X$. Then by the hypothesis,

$$_\rho Cl[f(F)] \subseteq f[{}_\rho Cl(F)] = f(F).$$

This implies, ${}_\rho Cl[f(F)] \subseteq f(F)$. Hence ${}_\rho Cl[f(F)] = f(F)$, that is, $f(F)$ is a closed set in Y. Hence f is $\mathcal{G}^\omega$-closed function. □

3. $\theta{-}\mathcal{G}^\omega$-CONTINUOUS FUNCTIONS

DEFINITION 3.1. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ of a grill topological space $(X, \tau, \mathcal{G})$ into a space (Y, ρ) is called *$\theta{-}\mathcal{G}^\omega$-continuous function* if for each $x \in X$ and each open set V in (Y, ρ) containing $f(x)$, there exists $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x such that $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V)$.

THEOREM 3.2. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is $\theta{-}\mathcal{G}^\omega$-continuous if and only if

$$_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(V)) \subseteq f^{-1}({}_\rho Cl(V))$$

for every open set V in (Y, ρ).

PROOF. Suppose that f is $\theta{-}\mathcal{G}^\omega$-continuous. Let V be any open set in of (Y, ρ). Let $x \notin f^{-1}({}_\rho Cl(V))$. Then $f(x) \notin {}_\rho Cl(V)$. Then $f(x) \in Y - {}_\rho Cl(V)$. Since $Y - {}_\rho Cl(V)$ is open set in (Y, ρ) containing x and f is $\theta{-}\mathcal{G}^\omega$-continuous then there exists $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x such that

$$f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(Y - {}_\rho Cl(V)).$$

This implies,

$$f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(Y - {}_\rho Cl(V)) = Y - {}_\rho Int({}_\rho Cl(V)).$$

Hence

$$f({}_{\mathcal{G}^\omega}Cl(U)) \cap {}_\rho Int({}_\rho Cl(V)) = \emptyset.$$

Since

$$V = {}_\rho Int(V) \subseteq {}_\rho Int({}_\rho Cl(V))$$

then $f({}_{\mathcal{G}^\omega}Cl(U)) \cap V = \emptyset$ and so $_{\mathcal{G}^\omega}Cl(U) \cap f^{-1}(V) = \emptyset$. Since U is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ containing x then $x \notin {}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(V))$. Hence

$$_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(V)) \subseteq f^{-1}({}_\rho Cl(V)).$$

Conversely, Let $x \in X$ be any point in X and V be any open set (Y, ρ) containing $f(x)$. Since

$$V \cap (Y - {}_\rho Cl(V)) = \emptyset$$

then

$$f(x) \notin {}_\rho Cl(Y - {}_\rho Cl(V)).$$

This implies,

$$x \notin f^{-1}[{}_\rho Cl(Y - {}_\rho Cl(V))].$$

Since $Y - {}_\rho Cl(V)$ is an open set in (Y, ρ) then by the hypothesis,

$$_{\mathcal{G}^\omega}Cl^\theta[f^{-1}(Y - {}_\rho Cl(V))] \subseteq f^{-1}[{}_\rho Cl(Y - {}_\rho Cl(V))].$$

Then

$$x \notin {}_{\mathcal{G}^\omega}Cl^\theta[f^{-1}(Y - {}_\rho Cl(V))].$$

Hence there is $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x such that

$$_{\mathcal{G}^\omega}Cl(U) \cap f^{-1}(Y - {}_\rho Cl(V)) = \emptyset.$$

This implies, $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V)$. Hence f is $\theta{-}\mathcal{G}^\omega$-continuous. □

THEOREM 3.3. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is $\theta{-}\mathcal{G}^\omega$-continuous if and only if

$$_{\mathcal{G}^\omega}Cl^\theta[X - f^{-1}({}_\rho Cl(V))] \subseteq X - f^{-1}(V)$$

for every open set V in (Y, ρ).

PROOF. Suppose that f is $\theta-\mathcal{G}^\omega$-continuous. Let V be any open set in of (Y,ρ). Let $x \notin X - f^{-1}(V)$. Then $f(x) \in V$. Since f is $\theta-\mathcal{G}^\omega$-continuous then there exists $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that

$$f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V).$$

This implies,

$${}_{\mathcal{G}^\omega}Cl(U) \subseteq f^{-1}({}_\rho Cl(V)).$$

Then

$${}_{\mathcal{G}^\omega}Cl(U) \cap [X - f^{-1}({}_\rho Cl(V))] = \emptyset.$$

Since U is a $\mathcal{G}^\omega$-open set in $(X,\tau,\mathcal{G})$ containing x then

$$x \notin {}_{\mathcal{G}^\omega}Cl^\theta[X - f^{-1}({}_\rho Cl(V))].$$

Hence

$${}_{\mathcal{G}^\omega}Cl^\theta[X - f^{-1}({}_\rho Cl(V))] \subseteq X - f^{-1}(V).$$

Conversely, let $x \in X$ be any point in X and V be any open set (Y,ρ) containing $f(x)$. Then $x \in f^{-1}(V)$, that is, $x \notin X - f^{-1}(V)$. then by the hypothesis,

$$x \notin {}_{\mathcal{G}^\omega}Cl^\theta[X - f^{-1}({}_\rho Cl(V))].$$

That is, there is there is $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that

$${}_{\mathcal{G}^\omega}Cl(U) \cap [X - f^{-1}({}_\rho Cl(V))] = \emptyset.$$

This implies, ${}_{\mathcal{G}^\omega}Cl(U) \subseteq f^{-1}({}_\rho Cl(V))$ and so $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V)$. Hence f is $\theta-\mathcal{G}^\omega$-continuous. □

THEOREM 3.4. For a function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$, the following properties are equivalent:

(1) f is $\theta-\mathcal{G}^\omega$-continuous.
(2) ${}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}({}_\rho Cl^\theta(B))$ for every subset $B \subseteq Y$.
(3) $f({}_{\mathcal{G}^\omega}Cl^\theta(A)) \subseteq {}_\rho Cl^\theta(f(A))$ for every subset $A \subseteq X$.

PROOF. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin f^{-1}({}_\rho Cl^\theta(B))$. Then $f(x) \notin {}_\rho Cl^\theta(B)$. Then there is an open set V in Y containing $f(x)$ such that ${}_\rho Cl(V) \cap B = \emptyset$. Since f is $\theta-\mathcal{G}^\omega$-continuous then there exists $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V)$. Then we have $f({}_{\mathcal{G}^\omega}Cl(U)) \cap B = \emptyset$. This implies, ${}_{\mathcal{G}^\omega}Cl(U) \cap f^{-1}(B) = \emptyset$. Hence $x \notin {}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B))$. That is,

$${}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}({}_\rho Cl^\theta(B)).$$

(2) ⇒ (1): Let $x \in X$ be any point in X and V be any open set (Y,ρ) containing $f(x)$. Since

$${}_\rho Cl(V) \cap (Y - {}_\rho Cl(V)) = \emptyset$$

then

$$f(x) \notin {}_\rho Cl^\theta(Y - {}_\rho Cl(V)).$$

This implies,

$$x \notin f^{-1}[{}_\rho Cl^\theta(Y - {}_\rho Cl(V))].$$

Since

$${}_\rho Cl^\theta(Y - {}_\rho Cl(V)) \subseteq Y$$

then by the hypothesis,

$$\begin{aligned} & {}_{\mathcal{G}^\omega}Cl^\theta[f^{-1}({}_\rho Cl^\theta(Y - {}_\rho Cl(V)))] \\ \subseteq\ & f^{-1}[{}_\rho Cl^\theta({}_\rho Cl^\theta(Y - {}_\rho Cl(V)))] \\ =\ & f^{-1}[{}_\rho Cl^\theta(Y - {}_\rho Cl(V))]. \end{aligned}$$

Then

$$x \notin {}_{\mathcal{G}^\omega}Cl^\theta[f^{-1}({}_\rho Cl^\theta(Y - {}_\rho Cl(V)))].$$

Hence there is $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that

$${}_{\mathcal{G}^\omega}Cl(U) \cap f^{-1}[{}_\rho Cl^\theta(Y - {}_\rho Cl(V))] = \emptyset.$$

This implies, $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V)$. Hence f is $\theta-\mathcal{G}^\omega$-continuous.
(2) ⇒ (3): Let A be any subset of X. Since $f(A) \subseteq Y$ then by the hypothesis,

$${}_{\mathcal{G}^\omega}Cl^\theta(A) \subseteq {}_{\mathcal{G}^\omega}Cl^\theta[f^{-1}(f(A))] \subseteq f^{-1}[{}_\rho Cl^\theta(f(A))].$$

This implies,

$$f({}_{\mathcal{G}^\omega}Cl^\theta(A)) \subseteq {}_\rho Cl^\theta(f(A)).$$

(3) ⇒ (2): Let B be any subset of Y. Since $f^{-1}(B) \subseteq X$ then by the hypothesis,

$$f[{}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B))] \subseteq {}_\rho Cl^\theta[f(f^{-1}(B))] \subseteq {}_\rho Cl^\theta(B).$$

This implies,

$${}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}({}_\rho Cl^\theta(B)).$$

□

THEOREM 3.5. A function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ is $\theta-\mathcal{G}^\omega$-continuous if and only if a function $g : (X,\tau,\mathcal{G}) \to (X \times Y, \tau \times \rho)$ is $\theta-\mathcal{G}^\omega$-continuous, where $g(x) = (x, f(x))$ for all $x \in X$.

PROOF. Suppose that g is $\theta-\mathcal{G}^\omega$-continuous. Let $x \in X$ be any point in X and V be any open set in (Y,ρ) containing $f(x)$. Then $X \times V$ is an open set in $(X \times Y, \tau \times \rho)$. Since g is $\theta-\mathcal{G}^\omega$-continuous there exists $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that

$$g({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_{\tau\times\rho}Cl(X \times V).$$

This implies,

$${}_{\mathcal{G}^\omega}Cl(U) \times f({}_{\mathcal{G}^\omega}Cl(U)) = g({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_{\tau\times\rho}Cl(X \times V)$$

$= X \times {}_\rho Cl(V)$. Then $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V)$. Hence f is $\theta-\mathcal{G}^\omega$-continuous.
Conversely, suppose that f is $\theta-\mathcal{G}^\omega$-continuous. Let $x \in X$ be any point in X and W be any open set in $(X \times Y, \tau \times \rho)$ containing $g(x)$. Then there are open sets $G \subseteq X$ and $V \subseteq Y$ such that

$$g(x) = (x, f(x)) \in G \times V \subseteq W.$$

Since f is $\theta-\mathcal{G}^\omega$-continuous there exists $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that

$$f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\rho Cl(V).$$

Let $H = U \cap G$. Then by Theorem (1.4), H is $\mathcal{G}^\omega$-open set in $(X,\tau,\mathcal{G})$ containing x. Hence we have

$$\begin{aligned} g({}_{\mathcal{G}^\omega}Cl(H)) &= {}_{\mathcal{G}^\omega}Cl(H) \times f({}_{\mathcal{G}^\omega}Cl(H)) \\ &\subseteq {}_{\mathcal{G}^\omega}Cl(U \cap G) \times f({}_{\mathcal{G}^\omega}Cl(U \cap G)) \\ &\subseteq {}_{\mathcal{G}^\omega}Cl(G) \times f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq {}_\tau Cl(G) \times {}_\rho Cl(V) \\ &= {}_{\tau\times\rho}Cl(G \times V) \subseteq {}_{\tau\times\rho}Cl(W). \end{aligned}$$

Hence g is $\theta-\mathcal{G}^\omega$-continuous. □

DEFINITION 3.6. A function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ of a grill topological space $(X,\tau,\mathcal{G})$ into a space (Y,ρ) is called *strongly $\theta-\mathcal{G}^\omega$-continuous function* if for each $x \in X$ and each open set V in (Y,ρ) containing $f(x)$, there exists $\mathcal{G}^\omega$-open set U in $(X,\tau,\mathcal{G})$ containing x such that $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq V$.

THEOREM 3.7. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is strongly $\theta{-}\mathcal{G}^\omega$-continuous if and only if $f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ for every open set V in (Y, ρ).

PROOF. Suppose that f is strongly $\theta{-}\mathcal{G}^\omega$-continuous. Let V be any open set in of (Y, ρ). We prove that $X - f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-closed set. Let $x \notin X - f^{-1}(V)$. Then $f(x) \in V$. Since f is strongly $\theta{-}\mathcal{G}^\omega$-continuous then there exists $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x such that $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq V$. This implies, ${}_{\mathcal{G}^\omega}Cl(U) \subseteq f^{-1}(V)$. Hence

$${}_{\mathcal{G}^\omega}Cl(U) \cap X - f^{-1}(V) = \emptyset.$$

Since U is $\mathcal{G}^\omega$-open set in $(X, \tau, \mathcal{G})$ containing x then

$$x \notin {}_{\mathcal{G}^\omega}Cl^\theta(X - f^{-1}(V)).$$

Hence

$${}_{\mathcal{G}^\omega}Cl^\theta(X - f^{-1}(V)) \subseteq X - f^{-1}({}_\rho Cl(V)).$$

Then $f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-open set.
Conversely, Let $x \in X$ be any point in X and V be any open set (Y, ρ) containing $f(x)$. Then by the hypothesis, $f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-open set, that is, $X - f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-closed set. Then

$$x \notin X - f^{-1}(V) = {}_{\mathcal{G}^\omega}Cl^\theta(X - f^{-1}(V)).$$

Hence there is $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x such that

$${}_{\mathcal{G}^\omega}Cl(U) \cap (X - f^{-1}(V)) = \emptyset.$$

This implies, $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq V$. Hence f is strongly $\theta{-}\mathcal{G}^\omega$-continuous. □

COROLLARY 3.8. A function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$ is strongly $\theta{-}\mathcal{G}^\omega$-continuous if and only if $f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$ for every closed set V in (Y, ρ).

THEOREM 3.9. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \rho)$, the following properties are equivalent:

(1) f is strongly $\theta{-}\mathcal{G}^\omega$-continuous.
(2) $f({}_{\mathcal{G}^\omega}Cl^\theta(A)) \subseteq {}_\rho Cl(f(A))$ for every subset $A \subseteq X$.
(3) ${}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}({}_\rho Cl(B))$ for every subset $B \subseteq Y$.

PROOF. (1) ⇒ (2): Let A be any subset of X. Suppose that $y \notin {}_\rho Cl(f(A))$. Then there is an open set V in Y containing y such that $f(x) = y$ and $V \cap f(A) = \emptyset$. Since f is strongly $\theta{-}\mathcal{G}^\omega$-continuous then there exists $\mathcal{G}^\omega$-open set U in $(X, \tau, \mathcal{G})$ containing x such that $f({}_{\mathcal{G}^\omega}Cl(U)) \subseteq V$. Then we have

$$f[{}_{\mathcal{G}^\omega}Cl(U) \cap A] \subseteq f({}_{\mathcal{G}^\omega}Cl(U)) \cap f(A) = \emptyset.$$

This implies, ${}_{\mathcal{G}^\omega}Cl(U) \cap A = \emptyset$. Hence $x \notin {}_{\mathcal{G}^\omega}Cl^\theta(A)$. That is, $y \notin f_{\mathcal{G}^\omega}Cl^\theta(A))$. Hence

$$f({}_{\mathcal{G}^\omega}Cl^\theta(A)) \subseteq {}_\rho Cl(f(A)).$$

(2) ⇒ (3): Let B be any subset of Y. Since $f^{-1}(B) \subseteq X$ then by the hypothesis,

$$f[{}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B))] \subseteq {}_\rho Cl[f(f^{-1}(B))] \subseteq {}_\rho Cl(B).$$

Hence

$${}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(B)) \subseteq f^{-1}({}_\rho Cl(B)).$$

(3) ⇒ (1): Let V be any open set in (Y, ρ). Since $Y - V$ is closed set in Y and by the hypothesis,

$$\begin{aligned} {}_{\mathcal{G}^\omega}Cl^\theta(X - f^{-1}(V)) &= {}_{\mathcal{G}^\omega}Cl^\theta(f^{-1}(Y - V)) \\ &\subseteq f^{-1}({}_\rho Cl(Y - V)) \\ &= f^{-1}(Y - V) = X - f^{-1}(V). \end{aligned}$$

Hence $X - f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-closed set in $(X, \tau, \mathcal{G})$. That is, $f^{-1}(V)$ is $\theta{-}\mathcal{G}^\omega$-i open set in $(X, \tau, \mathcal{G})$. Then by Theorem (3.7), f is strongly $\theta{-}\mathcal{G}^\omega$-continuous. □

THEOREM 3.10. Every strongly $\theta{-}\mathcal{G}^\omega$-continuous is $\mathcal{G}^\omega$-continuous.

PROOF. From Theorem (3.7) and the fact every $\theta{-}\mathcal{G}^\omega$-open set is $\mathcal{G}^\omega$-open set. □

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